

# On the linear quadratic problem for systems with time reversed Markov jump parameters and the duality with filtering of Markov jump linear systems

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**Abstract**—We study a class of systems whose parameters are driven by a Markov chain in reverse time. A recursive characterization for the second moment matrix, a spectral radius test for mean square stability and the formulas for optimal control are given. Our results are determining for the question: is it possible to extend the classical duality between filtering and control of linear systems (whose matrices are transposed in the dual problem) by simply adding the jump variable of a Markov jump linear system. The answer is positive provided the jump process is reversed in time.

## I. INTRODUCTION

In this note we study a class of systems whose parameters are driven by a time reversed Markov chain. Given a time horizon  $\ell$  and a standard Markov chain  $\{\eta(t), t = 0, 1, \dots\}$  taking values in the set  $\{1, 2, \dots, N\}$ , we consider the process

$$\theta(t) = \eta(\ell - t), \quad 0 \leq t \leq \ell, \quad (1)$$

and the system

$$\Phi : \begin{cases} x(t+1) &= A_{\theta(t)}x(t) + B_{\theta(t)}u(t), \\ x(0) &= x_0, \quad 0 \leq t \leq \ell - 1, \end{cases} \quad (2)$$

where, as usual,  $x$  represents the state variable of the system and  $u$  is the control variable. These systems may be encountered in real world problems, specially when a Markov chain interacts with the system parameters via a *first in last out* queue. An example consists of drilling sedimentary rocks whose layers can be modelled by a Markov chain from bottom to top as a consequence of their formation process. The first drilled layer is the last formed one. Another example is a DC-motor whose brush is grind by a machine subject to failures, leaving a series of imprecisions on the brush width that can be described by a Markov chain, so that the last failure will be the first to affect the motor collector depending on how the brush is installed.

One of the most remarkable features of system  $\Phi$  is that it provides a dual for optimal filtering of standard *Markov jump linear systems* (MJLS). In fact, if we consider a quadratic cost functional for system  $\Phi$  with linear state feedback, leading to an optimal control problem [4] that we call *time reversed Markov jump linear quadratic* problem (TRM-JLQP), then we show that the solution is identical to the gains of the *linear minimum mean square estimator*

(LMMSE) formulated in [8], [9], with time-reversed gains and transposed matrices. In perspective with existing duality relations, the one obtained here is a direct generalization of the well known relation between control and filtering of linear time varying systems as presented for instance in [10, Table 6.1], or also in [3], [16], [17] in different contexts. As for MJLS, the duality between control and filtering have been considered e.g. in [2], [5], [7], [8], [12], [13], [15], while purely in the context of standard MJLS, thus leading to more complex relations involving certain generalized coupled Riccati difference equations. Here, the duality follows naturally from the simple reversion of the Markov chain given in (1), with no extra assumptions nor complex constructions.

Another interesting feature of  $\Phi$  is that the variable  $\mathbb{E}\{x(t)x(t)' \cdot \mathbb{1}_{\{\theta(t)=i\}}\}$ , which is commonly used in the literature of MJLS, [1], [8], [11], [14], evolves along time  $t$  according to a *time-varying* linear operator, as shown in Remark 1, in a marked dissimilarity with standard MJLS. This motivated us to employ  $\mathfrak{X}(k)$ , the conditioned second moment of  $x(k)$ , leading to time-homogeneous operators.

The contents of this note are as follows. We present basic notation in Section II. In Section III we give the recursive equation describing  $\mathfrak{X}$ , which leads to a stability condition involving the spectral radius of a time-homogeneous linear operator. In Section IV, we formulate and solve the TRM-JLQ problem, following a proof method where we decompose  $\mathfrak{X}$  into two components as to handle  $\theta$  that are visited with zero probability. The duality with the LMMSE then follows in a straightforward manner, as presented in Section V. Concluding remarks are given in Section VI.

## II. NOTATION AND THE SYSTEM SETUP

Let  $\mathbb{R}^n$  be the  $n$ -dimensional euclidean space and  $\mathbb{R}^{m,n}$  be the space formed by real matrices of dimension  $m$  by  $n$ . We write  $\mathcal{C}^{m,n}$  to represent the Hilbert space composed of  $N$  real matrices, that is  $Y = (Y_1, \dots, Y_N) \in \mathcal{C}^{m,n}$ , where  $Y_i \in \mathbb{R}^{m,n}$ ,  $i = 1, \dots, N$ . The space  $\mathcal{C}^{n,m}$  equipped with the inner product  $\langle Y, Z \rangle = \sum_{i=1}^N \text{Tr}(Y_i' Z_i)$ , where  $\text{Tr}(\cdot)$  is the trace operator and the superscript  $'$  denotes the transpose, is a Hilbert space. The inner product induces the norm  $\|Y\| = \langle Y, Y \rangle^{1/2}$ . If  $n = m$ , we write simply  $\mathcal{C}^n$ . The mathematical operations involving elements of  $\mathcal{C}^{n,m}$ , are used in element-wise fashion, e.g. for  $Y$  and  $Z$  in  $\mathcal{C}^n$  we have  $YZ = (Y_1 Z_1, \dots, Y_N Z_N)$ , where  $Y_i Z_i$  is the usual

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matrix multiplication. Similarly, for a set of scalars  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathcal{C}^1$  we write  $\alpha Y = (\alpha_1 Y_1, \dots, \alpha_N Y_N)$ .

Regarding the system setup, it is assumed throughout the paper that  $x_0 \in \mathbb{R}^n$  is a random variable with zero mean satisfying  $\mathbb{E}\{x_0 x_0'\} = \Delta$ . We have  $x(t) \in \mathbb{R}^n$  and  $u(t) \in \mathbb{R}^m$ . The system matrices belong to given sets  $A, E \in \mathcal{C}^n$ ,  $B \in \mathcal{C}^{n,m}$ ,  $C \in \mathcal{C}^{s,n}$  and  $D \in \mathcal{C}^{s,m}$  with  $C_i' D_i = 0$  and  $D_i' D_i > 0$  for each  $i = 1, \dots, N$ . We write  $\pi_i(t) = \Pr(\theta(t) = i)$ , where  $\Pr(\cdot)$  is the probability measure;  $\pi(t)$  is considered as an element of  $\mathcal{C}^1$ , that is,  $\pi(t) = (\pi_1(t), \dots, \pi_N(t))$ .  $\pi_i$  stands for the limiting distribution of the Markov chain  $\eta$  when it exists, in such a manner that  $\pi_i = \lim_{\ell \rightarrow \infty} \pi_i(0)$ . Also, we denote by  $\mathcal{P} = [p_{ij}]$ ,  $i, j = 1, \dots, N$  the transition probability matrix of the Markov chain  $\eta$ , so that for any  $t = 1, \dots, \ell$ ,

$$\begin{aligned} \Pr(\theta(t-1) = j | \theta(t) = i) &= \\ &= \Pr(\eta(\ell-t+1) = j | \eta(\ell-t) = i) = p_{ij}. \end{aligned}$$

No additional assumption is made on the Markov chain, yielding a rather general setup that includes periodic chains, important for the duality relation given in Remark 2. We shall deal with linear operators  $\mathcal{U}_Z, \mathcal{V}_Z, \mathcal{D} : \mathcal{C}^n \rightarrow \mathcal{C}^n$ . We write the  $i$ -th element of  $\mathcal{U}_Z(Y)$  by  $\mathcal{U}_{Z,i}(Y)$  and similarly for the other operators. For each  $i = 1, \dots, N$ , we define:

$$\begin{aligned} \mathcal{U}_{Z,i}(Y) &= \sum_{j=1}^N p_{ij} Z_j Y_j Z_j' \\ \mathcal{V}_{Z,i}(Y) &= Z_i' \mathcal{D}_i(Y) Z_i, \\ \mathcal{D}_i(Y) &= \sum_{j=1}^N p_{ji} Y_j. \end{aligned} \quad (3)$$

### III. PROPERTIES OF SYSTEM $\Phi$

Let  $\mathbb{E}\{\cdot\}$  be the expected value of a random variable. We consider the *conditioned second moment* of  $x(t)$  defined by

$$\mathfrak{X}_i(t) = \mathbb{E}\{x(t)x(t)' | \theta(t) = i\}, \quad t = 0, 1, \dots \quad (4)$$

*Lemma 3.1:* Consider the system  $\Phi$  with  $u(t) = 0$  for each  $t$ . The conditioned second moment  $\mathfrak{X}(t) \in \mathcal{C}^n$  is given by  $\mathfrak{X}(0) = (\Delta, \dots, \Delta)$  and

$$\mathfrak{X}(t+1) = \mathcal{U}_A(\mathfrak{X}(t)), \quad t = 0, 1, \dots, \ell-1. \quad (5)$$

*Proof:* For a fixed, arbitrary  $i \in \{0, \dots, N\}$ , note that

$$\mathfrak{X}_i(0) = \mathbb{E}\{x_0 x_0' | \theta(0) = i\} = \mathbb{E}\{x_0 x_0'\} = \Delta.$$

From (2), (4) and the total probability law we obtain:

$$\begin{aligned} \mathfrak{X}_i(t+1) &= \mathbb{E}\{A_{\theta(t)} x(t) x(t)' A_{\theta(t)}' | \theta(t+1) = i\} \\ &= \sum_{j=1}^N \mathbb{E}\{A_{\theta(t)} x(t) x(t)' A_{\theta(t)}' \cdot \mathbb{1}_{\theta(t)=j} | \theta(t+1) = i\}. \end{aligned} \quad (6)$$

In order to compute the right hand side of (6), we need the following standard Markov chain property: for any function

$\Gamma : \theta(0), \dots, \theta(t) \rightarrow \mathbb{R}^{n,n}$  we have

$$\begin{aligned} &\mathbb{E}\{\Gamma(\theta(0), \dots, \theta(t)) \cdot \mathbb{1}_{\theta(t)=j} | \theta(t+1) = i\} \\ &= \mathbb{E}\{\Gamma(\eta(\ell), \dots, \eta(\ell-t)) \cdot \mathbb{1}_{\eta(\ell-t)=j} | \eta(\ell-t-1) = i\} \\ &= \mathbb{E}\{\Gamma(\eta(\ell), \dots, \eta(\ell-t)) | \eta(\ell-t) = j, \eta(\ell-t-1) = i\} \\ &\quad \cdot \text{Prob}(\eta(\ell-t) = j | \eta(\ell-t-1) = i) \\ &= \mathbb{E}\{\Gamma(\eta(\ell), \dots, \eta(\ell-t)) | \eta(\ell-t) = j\} p_{ij} \\ &= \mathbb{E}\{\Gamma(\theta(0), \dots, \theta(t)) | \theta(t) = j\} p_{ij}, \end{aligned}$$

then by replacing  $\Gamma$  with  $A_{\theta(t)} x(t) x(t)' A_{\theta(t)}'$  and applying the above in (6) yields

$$\mathfrak{X}_i(t+1) = \sum_{j=1}^N p_{ij} A_j \mathfrak{X}_j(t) A_j' = \mathcal{U}_{A,i}(\mathfrak{X}(t)),$$

which completes the proof.  $\blacksquare$

*Remark 1:* Let  $W(t) \in \mathcal{C}^n$ ,  $t = 0, \dots, \ell$  be given by

$$W_i(t) = \mathbb{E}\{x(t)x(t)' \cdot \mathbb{1}_{\theta(t)=i}\}. \quad (7)$$

This variable is commonly encountered in the majority of papers dealing with (standard) MJLS. However, calculations similar to that in Lemma 3.1 lead to

$$W_i(t+1) = \sum_{j=1}^N p_{ij} \frac{\pi_i(t+1)}{\pi_j(t)} A_j W_j(t) A_j'. \quad (8)$$

Note that the Markov chain measure appears explicitly, leading to a time-varying mapping from  $W(t)$  to  $W(t+1)$ . The only exception is when the Markov chain is *reversible*, in which case the facts that  $\pi_j p_{ji} = \pi_i p_{ij}$  and that the Markov chain starts with the invariant measure (by definition) yield

$$p_{ij} \frac{\pi_i(t+1)}{\pi_j(t)} = p_{ij} \frac{\pi_i}{\pi_j} = p_{ji},$$

in which case  $W$  evolves exactly as in a standard MJLS.

The following notion is adapted from [8, Chapter 3].

*Definition 3.1:* We say that the system  $\Phi$  with  $u(t) = 0$  is *mean square stable* (MS-stable), whenever

$$\lim_{\ell \rightarrow \infty} \mathbb{E}\{\|x(\ell)\|^2\} = 0.$$

This is equivalent to say that the variable  $\mathfrak{X}(\ell)$  converges to zero as  $\ell$  goes to infinity, leading to the following result.

*Theorem 3.1:* The system  $\Phi$  with  $u(t) = 0$  is MS-stable if and only if the spectral radius of  $\mathcal{U}_A$  is smaller than one.

### IV. THE TRM-JLQ PROBLEM

Let the output variable  $y$  given by  $y(\ell) = E_{\theta(\ell)} x(\ell)$  and

$$y(t) = C_{\theta(t)} x(t) + D_{\theta(t)} u(t), \quad t = 0, \dots, \ell-1.$$

The TRM-JLQ consists of minimizing the mean square of  $y$  with  $\ell$  stages, as usual in jump linear quadratic problems,

$$\min_{u(0), \dots, u(\ell-1)} \mathbb{E} \left\{ \sum_{t=0}^{\ell} \|y(t)\|^2 \right\}. \quad (9)$$

Regarding the information structure of the problem, we assume that  $\theta(t)$  is available to the controller, that the control is in linear state feedback form,

$$u(t) = K_{\theta(t)}(t) x(t), \quad t = 0, 1, \dots, \ell-1, \quad (10)$$

where  $K(t) \in \mathcal{C}^{m,n}$  is the decision variable, and that one should be able to compute the sequence  $K(0), \dots, K(\ell-1)$  prior to the system operation, that is,  $K(t)$  is not a function of the observations  $(x(s), \theta(s))$ ,  $0 \leq s \leq \ell$ . The conditioned second moment  $\mathfrak{X}$  for the closed loop system is of much help in obtaining the solution. The recursive formula for  $\mathfrak{X}$  follows by a direct adaptation of Lemma 3.1, by replacing  $A \in \mathcal{C}^n$  with its closed loop version

$$A_i(t) = A_i + B_i K_i(t). \quad (11)$$

*Lemma 4.1:* The conditioned second moment  $\mathfrak{X}(t) \in \mathcal{C}^n$  is given by  $\mathfrak{X}(0) = (\Delta, \dots, \Delta)$  and

$$\mathfrak{X}(t+1) = \mathcal{U}_{A(t)}(\mathfrak{X}(t)), \quad t = 0, 1, \dots, \ell-1. \quad (12)$$

In what follows, for brevity we denote

$$\begin{aligned} \mathcal{Q}(\ell) &= \pi(\ell) E' E, \\ \mathcal{Q}(t) &= C' C + K(t)' D' D K(t), \quad t = 0, \dots, \ell-1. \end{aligned}$$

*Lemma 4.2:* The TRM-JLQ problem can be formulated as

$$\min_{K(0), \dots, K(\ell-1)} \left\{ \sum_{t=0}^{\ell} \langle \pi(t) \mathcal{Q}(t), \mathfrak{X}(t) \rangle \right\}. \quad (13)$$

*Proof:* The mean square of the terminal cost,  $y(\ell)$  is:

$$\begin{aligned} \mathbb{E}\{\|y(\ell)\|^2\} &= \sum_{i=1}^N \mathbb{E}\{x(\ell)' E'_{\theta(\ell)} E_{\theta(\ell)} x(\ell) \cdot \mathbf{1}_{\theta(\ell)=i}\} \\ &= \sum_{i=1}^N \text{Tr}(\pi_i(\ell) E' E_i \mathfrak{X}_i(\ell)) = \langle \pi(\ell) E' E, \mathfrak{X}(\ell) \rangle. \end{aligned} \quad (14)$$

Now, by a calculation similar as above leads to

$$\mathbb{E}\{\|y(t)\|^2\} = \langle \pi(t) (C' C + K(t)' D' D K(t)), \mathfrak{X}(t) \rangle. \quad (15)$$

Substituting (14) and (15) into (9) we obtain (13). ■

Let us denote the gains attaining (13) by  $K^{\text{op}}(t)$ . From a dynamic programming standpoint, we introduce value functions  $V^t : \mathcal{C}^n \rightarrow \mathbb{R}$  by:  $V^\ell = \langle \pi(\ell) \mathcal{Q}(\ell), \mathfrak{X}(\ell) \rangle$  and for  $t = \ell-1, \ell-2, \dots, 0$ ,

$$V^t(\mathfrak{X}) = \min_{K(t), \dots, K(\ell-1)} \left\{ \sum_{\tau=t}^{\ell} \langle \pi(\tau) \mathcal{Q}(\tau), \mathfrak{X}(\tau) \rangle \right\}, \quad (16)$$

where  $\mathfrak{X}(t) = \mathfrak{X}$  and  $\mathfrak{X}(\tau)$ ,  $\tau = t+1, \dots, \ell$ , satisfies (12).

*Theorem 4.1:* Define  $P(t) \in \mathcal{C}^n$  and  $M(t) \in \mathcal{C}^{m,n}$ ,  $t = 0, \dots, \ell-1$ , as follows. Let  $P(\ell) = \pi(\ell) E' E$  and for each  $t = \ell-1, \dots, 0$  and  $i = 1, \dots, N$ , compute: if  $\pi_i(t) = 0$ ,

$$M_i(t) = 0 \quad \text{and} \quad P_i(t) = 0,$$

else (if  $\pi_i(t) > 0$ ),

$$\begin{aligned} R_i(t) &= (B_i' \mathcal{D}_i(P(t+1)) B_i + \pi_i(t) D_i' D_i), \\ M_i(t) &= R_i(t)^{-1} B_i' \mathcal{D}_i(P(t+1)) A_i, \end{aligned} \quad (17)$$

$$\begin{aligned} P_i(t) &= \pi_i(t) C_i' C_i + A_i' \mathcal{D}_i(P(t+1)) A_i \\ &\quad - A_i' \mathcal{D}_i(P(t+1)) B_i R_i(t)^{-1} B_i' \mathcal{D}_i(P(t+1)) A_i. \end{aligned} \quad (18)$$

Then,

$$V^t(\mathfrak{X}) = \langle P(t), \mathfrak{X} \rangle \quad (19)$$

and  $K^{\text{op}}(t) = M(t)$ ,  $t = 0, \dots, \ell$ .

*Proof:* We apply the dynamic programming approach for the costs defined in (16) and the system in (12), whose state is the variable  $\mathfrak{X} := \mathfrak{X}(t)$ . It can be checked that  $\mathcal{U}$  is the adjoint operator of  $\mathcal{V}$ , and consequently

$$\begin{aligned} V^t(\mathfrak{X}) &= \min_{K(t)} \langle \pi(t) \mathcal{Q}(t), \mathfrak{X} \rangle + \langle P(t+1), \mathcal{U}_{A(t)}(\mathfrak{X}) \rangle, \\ &= \min_{K(t)} \langle \pi(t) \mathcal{Q}(t) + A(t)' \mathcal{D}(P(t+1)) A(t), \mathfrak{X} \rangle. \end{aligned} \quad (20)$$

Let us decompose  $\mathfrak{X}$  as follows; let the set of states  $\theta$  having zero probability of being visited at time  $t$  be denoted by

$$\mathcal{N}_t = \{i : \pi_i(t) = 0\}.$$

We write  $\mathfrak{X} = \mathfrak{X}^N + \mathfrak{X}^P$ , where  $\mathfrak{X}^N$  is such that  $\mathfrak{X}_i^N = 0$  for any  $i \notin \mathcal{N}_t$ , and in a similar fashion  $\mathfrak{X}_i^P = 0$  for  $i \in \mathcal{N}_t$ . We now show that the term

$$\langle \pi(t) \mathcal{Q}(t) + A(t)' \mathcal{D}(P(t+1)) A(t), \mathfrak{X}^N \rangle$$

is zero irrespectively of  $K(t)$ . First, note that for  $i \in \mathcal{N}_t$  we have  $\pi_i(t) \mathcal{Q}_i(t) \mathfrak{X}_i^N = 0$ . Second, for  $i \in \mathcal{N}_t$  one can check that  $\pi_j(t+1) = 0$  for all  $j$  such that  $p_{ji} > 0$ , so that  $P_j(t+1) = 0$ . This yields  $\mathcal{D}_i(P(t+1)) = 0$ . Bringing these facts together and recalling that by construction  $\mathfrak{X}_i^N = 0$  for all  $i \notin \mathcal{N}_t$ , we evaluate

$$\langle \pi(t) \mathcal{Q}(t) + A(t)' \mathcal{D}(P(t+1)) A(t), \mathfrak{X}^N \rangle = 0. \quad (21)$$

By substituting (21) into (20) we write

$$\begin{aligned} V^t(\mathfrak{X}) &= \min_{K(t)} \langle \pi(t) \mathcal{Q}(t) + A(t)' \mathcal{D}(P(t+1)) A(t), \mathfrak{X}^P \rangle. \\ &= \min_{K(t)} \sum_{\{i \notin \mathcal{N}_t\}} \text{Tr}((\pi_i(t) \mathcal{Q}_i(t) \\ &\quad + A_i(t)' \mathcal{D}_i(P(t+1)) A_i(t)) \mathfrak{X}_i^P). \end{aligned} \quad (22)$$

By expanding some terms and after some algebra to complete the squares, we have

$$\begin{aligned} V^t(\mathfrak{X}) &= \min_{K(t)} \sum_{\{i \notin \mathcal{N}_t\}} \text{Tr}((\pi_i(t) C_i' C_i + A_i' \mathcal{D}_i(P(t+1)) A_i \\ &\quad + (K_i(t) - O_i(t))' (B_i' \mathcal{D}_i(P(t+1)) B_i + \pi_i(t) D_i' D_i) \\ &\quad \cdot (K_i(t) - O_i(t)) - O_i(t)' (B_i' \mathcal{D}_i(P(t+1)) B_i \\ &\quad + \pi_i(t) D_i' D_i) O_i(t)) \mathfrak{X}_i^P), \end{aligned} \quad (23)$$

where  $O_i(t) = M_i(t)$  as given in (17). This makes clear that the minimal cost is achieved by setting  $K_i^{\text{op}}(t) = M_i(t)$ ,  $i \notin \mathcal{N}_t$ . Now, by replacing  $K_i(t)$  with  $K_i^{\text{op}}(t)$  in (23),

$$V^t(\mathfrak{X}) = \sum_{\{i \notin \mathcal{N}_t\}} \text{Tr}(P_i(t) \mathfrak{X}_i^P)$$

with  $P_i(t)$  as given in (18),  $i \notin \mathcal{N}_t$ . Finally, by choosing  $P_i(t) = 0$ ,  $i \in \mathcal{N}_t$ , we write

$$\begin{aligned} V^t(\mathfrak{X}) &= \sum_{\{i \notin \mathcal{N}_t\}} \text{Tr}(P_i(t) \mathfrak{X}_i^P) + \sum_{\{i \in \mathcal{N}_t\}} \text{Tr}(P_i(t) \mathfrak{X}_i^N), \\ &= \langle P(t), \mathfrak{X}^P \rangle + \langle P(t), \mathfrak{X}^N \rangle = \langle P(t), \mathfrak{X} \rangle, \end{aligned}$$

which completes the proof. ■

## V. THE DUALITY BETWEEN THE TRM-JLQ AND THE LMMSE FOR STANDARD MJLS

We consider the LMMSE for standard MJLS as presented in [6], [9]. The problem consists of finding the sequence of sets of gains  $K^f(t)$ ,  $t = 0, \dots, \ell$ , that minimizes the covariance of the estimation error  $\tilde{z}(t) = \hat{z}(t) - z(t)$  when the estimate is given by a Luenberger observer in the form

$$\hat{z}(t+1) = A_{\eta(t+1)}\hat{z}(t) + K_{\eta(t+1)}^f(t)(y(t) - L_{\eta(t)}\hat{z}(t)),$$

where  $y(t)$  is the output of the MJLS

$$\begin{cases} z(t+1) &= F_{\eta(t+1)}z(t) + G_{\eta(t+1)}\omega(t) \\ y(t) &= L_{\eta(t+1)}z(t) + H_{\eta(t+1)}\omega(t) \\ z(0) &= z_0, \end{cases} \quad (24)$$

and  $\omega(t)$  and  $z_0$  are i.i.d. random variables satisfying  $\mathbb{E}\{\omega(t)\} = 0$ ,  $\mathbb{E}\{\omega(t)\omega(t)'\} = I$  and  $\mathbb{E}\{z_0 z_0'\} = \Sigma$ . Moreover, it is assumed that  $L_i H_i' = 0$  and  $H_i H_i' > 0$ . We write  $v_i(t) = \text{Prob}(\eta(t) = i)$ ,  $i = 1, \dots, N$ , so that it is the time-reverse of  $\pi$ ,  $v(t) = \pi(\ell - t)$ . Note that we are considering the same problem as in [6], [9], though our notation is slightly different: here we assume that  $(y(t), \eta(t+1))$  is available for the filter to obtain  $\hat{z}(t+1)$  and the system matrices are indexed by  $\eta(t+1)$ , while in the standard formulation  $(y(t), \eta(t))$  are observed at time  $t$  and the system matrices are indexed by  $\eta(t)$ . This “time shifting” in  $\eta$  avoids a cluttering in the duality relation. Along the same line, instead of writing the filter gains as a function of the variable  $Y_i(t) = \mathbb{E}\{\tilde{z}(t)\tilde{z}(t)' \cdot \mathbb{1}_{\{\eta(t+1)=i\}}\}$ , given by the coupled Riccati difference equation [9, Equation 24]

$$Y_i(t+1) = \sum_{j=1}^N p_{ji} \{ F_j Y_j(t) F_j' + v_j(t) G_j G_j' - F_j Y_j(t) L_j' \cdot (L_j Y_j(t) L_j' + v_j(t) H_j H_j')^{-1} L_j Y_j(t) F_j' \}$$

whenever  $v_i(t) > 0$  and  $Y_i(t+1) = 0$  otherwise, in this note we use the variable  $S_i(t) = \mathbb{E}\{\tilde{z}(t)\tilde{z}(t)' \cdot \mathbb{1}_{\{\eta(t)=i\}}\}$  defined in [7], leading to  $Y(t) = \mathcal{D}(S(t))$ . Replacing this in the above equation, after some algebraic manipulation one obtains [7, Equation 8]:

$$S_i(t+1) = v_i(t) G_i G_i' + F_i \mathcal{D}_i(S(t)) F_i' - F_i \mathcal{D}_i(S(t)) L_i' \cdot (L_i F_i(S(t)) L_i' + v_i(t) H_i H_i')^{-1} L_i \mathcal{D}_i(S(t)) F_i', \quad (25)$$

whenever  $v_i(t) > 0$  and  $S_i(t+1) = 0$  otherwise, with initial condition  $S_i(0) = \mathbb{E}\{\tilde{z}(0)\tilde{z}(0)' \cdot \mathbb{1}_{\{\eta(0)=i\}}\} = v_i(0)\Sigma$ . The optimal gains are given for  $t = 0, \dots, \ell$  by

$$K_i^f(t) = F_i \mathcal{D}_i(S(t)) L_i' (L_i \mathcal{D}_i(S(t)) L_i' + v_i(t) H_i H_i')^{-1} \quad (26)$$

whenever  $v_i(t) > 0$  and  $K_i^f(t) = 0$  otherwise. The duality relations between the filtering and control problems are now evident by direct comparison between (18) and (25).  $F_i, L_i, G_i$  and  $H_i$  are replaced with  $A_i', B_i', C_i'$  and  $D_i'$ , respectively. Moreover, comparing the initial conditions of the coupled Riccati difference equations, we see  $\Sigma$  replaced with  $E'E$ . Also, we note that  $P(0), P(1), \dots, P(\ell)$  are equivalent to  $S(\ell), S(\ell-1), \dots, S(0)$ , with a similar relation

for the gains  $K_i^f$  and  $K_i^{\text{op}}$ . The Markov chains driving the filtering and control systems are time-reversed one to each other.

*Remark 2:* Time-varying parameters can be included both in standard MJLS and in  $\Phi$  by augmenting the Markov state as to describe the pair  $(\theta, t)$ ,  $1 \leq \theta \leq N$ ,  $0 \leq t \leq \ell$ , and considering a suitable matrix  $P$  of higher dimension  $N \times (\ell + 1)$ . Although this reasoning leads to a matrix  $P$  of high dimension, periodic and sparse, it is useful to make clear that our results are readily adaptable to plants whose matrices are in the form  $A_{\theta(t)}(t)$ . Either by this reasoning or by re-doing all computations given in this note for time-varying plants, we obtain the following generalization of [10, Table 6.1].

FILTERING of MJLS	CONTROL of $\Phi$
$F_i(t)$	$A_i'(t)$
$L_i(t)$	$B_i'(t)$
$G_i(t)$	$C_i'(t)$
$H_i(t)$	$D_i'(t)$
$K_i^f(t)$	$K_i^{\text{op}'}(\ell - t)$
$S_i(0) = v_i(0)\Sigma$	$P_i(\ell) = \pi_i(\ell)E_i'E_i$
$S_i(t)$	$P_i(\ell - t)$
$\eta_i(t)$	$\theta_i(\ell - t)$

TABLE I  
SUMMARY OF THE FILTERING/CONTROL DUALITY.  $t = 0, \dots, \ell$ .

## VI. CONCLUDING REMARKS

We have presented an operator theory characterization of the conditional second moment  $\mathfrak{X}$ , an MS stability test and formulas for the optimal control of system  $\Phi$ . The results have exposed some interesting relations with standard MJLS. For system  $\Phi$  it is fruitful to use the *true* conditional second moment  $\mathfrak{X}$  whereas for standard MJLS one has to resort to the variable  $W$  given in (7) to obtain a recursive equation similar to the ones expressed in the Lemmas 3.1 and 4.1. Moreover, these classes of systems are equivalent if and only if the Markov chain is revertible, as indicated in Remark 1. The solution of the TRM-JLQ problem is given in Theorem 4.1 in the form of a coupled Riccati equation that can be computed backwards prior to the system operation, as usual in linear quadratic problems for linear systems. The result beautifully extends the classic duality between filtering and control into the relations expressed in Table 1.

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